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LETTER TO THE EDITOR

Enumeration of directed site animals on two-dimensional lattices

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Abstract. We study the problem of directed site animals on the square, triangular and hexagonal lattices. We propose closed form expressions for $A(s)$, the number of animals of size s, on the square and triangular lattices. These expressions have been checked for $s \leq 33$ and $s \leq 10$ for the square and triangular lattices respectively by explicit enumeration. They imply that $A(s)$ varies as $\lambda^s s^{-\theta}$ for large s, where $\lambda = 3$ for the square lattice, and $\lambda = 4$ for the triangular, and $\theta = \frac{1}{2}$ for both. For the hexagonal lattice, we have found A(s) for $s \leq 48$, and our results are consistent with $\lambda = 2.0252 \pm 0.0005$ and $\theta = \frac{1}{2}$.

The problem of enumeration of animals of a given size is an old one (Harary 1967, Domb 1976, Klarner 1981). It corresponds to the $p \to 0$ limit of the percolation problem (Duarte 1978, 1979, Stauffer 1979) and is also related to the study of branched polymers in the dilute limit (Lubensky and Isaacson 1978a, b, 1979). Closed form expressions for the number of animals in terms of their size are known only for graphs with no cycles (Fisher and Essam 1961, Harary *et al* 1975), though some critical indices are known in two and three dimensions (Parisi and Sourlas 1981).

In this Letter, we enumerate animals defined on directed lattices. The problem is closely related to that of directed percolation (Cardy and Sugar 1980, Kinzel and Yeomans 1981). We study directed *site* animals on the acyclic directed square, triangular and the hexagonal lattices (figure 1). The corresponding directed site-percolation problem has been studied earlier by Bishir (1963), and is believed to lie in the same universality class as the directed bond-percolation.

A directed animal is defined as a set of lattice sites $\mathscr{A}$ such that each site $\alpha \in \mathscr{A}$ is reachable from a given fixed site (say the origin) by directed paths such that all

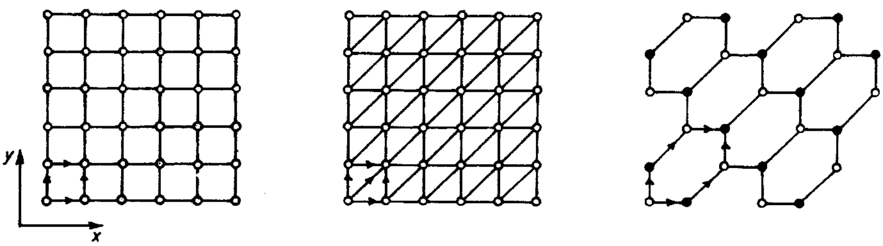

Figure 1. The orientations of the bonds in the directed square, triangular and hexagonal lattices considered in the text are shown in (a), (b) and (c) respectively. The full and open circles in (c) denote sites belonging to sublattices S_1 and S_2 respectively.

sites lying on the path are in $\mathscr{A}$. Note that the origin belongs to $\mathscr{A}$. The size of an animal $\mathscr{A}$ is the number of sites in $\mathscr{A}$ and is denoted by $|\mathscr{A}|$. The perimeter of $\mathscr{A}$ is defined as the number of points not in $\mathscr{A}$ which can be reached from some point in $\mathscr{A}$ by a single directed bond.

We define the generating function $G(x, y)$ as the sum of weights of all animals, the weight of an animal of size s and perimeter t being $x^s y^t$.

$$G(x, y) = \sum_{\mathscr{A}} x^s y^t = \sum_{s,t} A(s, t)\, x^s y^t \tag{1}$$

where $A(s, t)$ is the number of animals of size s and perimeter t.

$G(zp, 1-p)$ is the generating function for the cluster size distribution function of the corresponding site percolation problem,

$$G(zp, 1-p) = \sum_s z^s P_s(p), \tag{2}$$

where $P_s(p)$ is the probability that a cluster defined with the origin as source contains s sites. Here p is the concentration of occupied sites. $G(x, y=1)$ defines the generating function of the site animal problem

$$G(x, y=1) = \sum_s A(s) x^s \tag{3}$$

where $A(s) = \Sigma_t A(s, t)$ is the total number of site animals of size s. As in the case of undirected animals, for large s, $A(s)$ is expected to have the asymptotic form

$$A(s) \sim C\lambda^s s^{-\theta}. \tag{4}$$

Here C and λ are constants, different for different lattices, and θ is a critical exponent.

Let the generating functions for the square and triangular lattices be denoted by $G^{\mathrm{sq}}(x, y)$ and $G^{\mathrm{tr}}(x, y)$. On the hexagonal lattice, there are two sublattices S_1 and S_2. Sites on sublattice S_1 have two incoming bonds, while those on sublattice S_2 have only one incoming bond. We define two generating functions $G_1^{\mathrm{hex}}(x, y)$ and $G_2^{\mathrm{hex}}(x, y)$, depending on whether the source lies on S_1 or S_2. Clearly

$$G_1^{\mathrm{hex}}(x, y) = x[1 + G_2^{\mathrm{hex}}(x, y)]. \tag{5}$$

S_1 and S_2 are isomorphic to the square lattice, and with any animal $\mathscr{A}$ on the hexagonal lattice is associated an animal $\mathscr{A}'$ (of size s' and perimeter t') on S_2. To find $G_2^{\mathrm{hex}}(x, y)$, we first sum over the weights of all animals $\mathscr{A}$ consistent with a given $\mathscr{A}'$, and then sum over all $\mathscr{A}'$. If a site other than the origin is in $\mathscr{A}'$, the S_1 site leading to it must be in $\mathscr{A}$. Either a perimeter site of $\mathscr{A}'$ is a perimeter site of $\mathscr{A}$ (weight xy), or the S_1 site leading to it is a perimeter site of $\mathscr{A}$ (weight y). Thus the sum of all animals $\mathscr{A}$ consistent with a given $\mathscr{A}'$ of size s' and perimeter t' is $x^{2s'-1}(y+xy)^{t'}$. It follows that

$$G_2^{\mathrm{hex}}(x, y) = (1/x) G^{\mathrm{sq}}(x^2, y + xy). \tag{6}$$

This implies that the critical percolation probability for the directed site problem on the square lattice is the square of the value on the hexagonal lattice.

It is useful to consider animals with more than one point as sources. Let $\mathscr{R} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be a set of source points whose cartesian coordinates are (x_i, y_i). An animal $\mathscr{A}$ with source $\mathscr{R}$ is a set of points such that each $\alpha \in \mathscr{A}$ is reachable from at least one of the points in $\mathscr{R}$ using directed paths which do not visit any site outside $\mathscr{A}$. The source $\mathscr{R}$ is a subset of the animal $\mathscr{A}$. Let $A_{\mathscr{R}}(s)$ be the number of animals having s sites and generated from the source set $\mathcal{R}$. We shall be concerned only with those sets $\mathcal{R}$ where all the source sites lie on a line $x+y=$ constant (figure 1). Let $n_{\mathcal{R}}$ be the number of sites in $\mathcal{R}$. We define the length of $\mathcal{R}$ to be $(x_{\max}-x_{\min}+1)$ where $x_{\max}$ and $x_{\min}$ are the maximum and minimum x-coordinates of the source sites. We call the set $\mathcal{R}$ compact if its length is equal to $n_{\mathcal{R}}$. It is quite easy to write down relations between $A_{\mathcal{R}}(s)$ with different source sets $\mathcal{R}$. Consider, for definiteness, the square lattice. Let $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}'_1$ and $\mathcal{R}_2$ denote the source sets $\{(0,0)\}$, $\{(1,0)\}$, $\{(0,1)\}$ and $\{(1,0),(0,1)\}$ respectively. Then, clearly, for all $s>1$

$$A_{\mathcal{R}_0}(s)=A_{\mathcal{R}_1}(s-1)+A_{\mathcal{R}'_1}(s-1)+A_{\mathcal{R}_2}(s-1). \tag{7}$$

As $A_{\mathcal{R}_0}(s)=A_{\mathcal{R}_1}(s)=A_{\mathcal{R}'_1}(s)$ by translational invariance, we obtain

$$A_{\mathcal{R}_0}(s)=2A_{\mathcal{R}_0}(s-1)+A_{\mathcal{R}_2}(s-1). \tag{8}$$

Thus $A_{\mathcal{R}_0}(s)$ (same as $A(s)$ defined earlier) can be determined for all s, if $A_{\mathcal{R}_2}(s)$ are known. These, in turn, can be expressed in terms of animals with fewer sites. For example, if $\mathcal{R}_3=\{(2,0),(0,2)\}$ and $\mathcal{R}'_3=\{(2,0),(0,2),(1,1)\}$, we obtain using translational invariance

$$A_{\mathcal{R}_2}(s)=3A_{\mathcal{R}_1}(s-2)+2A_{\mathcal{R}_2}(s-2)+A_{\mathcal{R}_3}(s-2)+A_{\mathcal{R}'_3}(s-2). \tag{9}$$

We also notice that for all s

$$A_{\mathcal{R}_3}(s)=A_{\mathcal{R}'_3}(s+1). \tag{10}$$

Here $\mathcal{R}'_3$ is a compact source but $\mathcal{R}_3$ is not.

We have written a computer program which combines such recursions with explicit cluster counting for small s. The required recursion relations are generated by the program itself. Animals up to size 33 on the square lattice and up to size 48 on the hexagonal lattice were generated, each in about 1 hour of CPU time, on the CYBER 170–730 computer. For the triangular lattice, only straightforward generation was employed, and in about the same time only animals up to size 10 were generated. Our results are displayed in table 1.

The recursion relations used in the cluster enumeration involve both compact and non-compact sources. For a given length of $\mathcal{R}$, there are many non-compact sources but only one compact source. Let N_r^s be the number of animals of size s from a compact source of length r. Clearly $A(s)=N_1^s$. From our data, we noticed a rather non-trivial relation between these numbers N_r^s. For the square lattice, the relation is

$$N_r^s=N_{r-1}^{s-1}+N_r^{s-1}+N_{r+1}^{s-1}. \tag{11}$$

This has been verified for all $r+s\leqslant 34$ and $1\leqslant r\leqslant 10$. For the triangular lattice, the relation is

$$N_r^s=N_{r-1}^{s-1}+2N_r^{s-1}+N_{r+1}^{s-1}. \tag{12}$$

This has been verified only for $r=1$.

Equations (11) and (12) hold for $r=1$ also, with the definition $N_0^s=N_1^s$. The relations are important observations as, if we assume that these hold exactly for all r and s, they can be solved to give closed form expressions for N_r^s. We use the boundary conditions that $N_r^1=\delta_{r0}+\delta_{r1}$ and $N_0^s=N_1^s$, for all r and s.

Table 1. The number of directed site animals on some two-dimensional lattices determined by cluster counting using a computer. For the hexagonal lattice, the source site is assumed to be on the sublattice S_2.

n	Triangular	Square	Hexagonal
1	1	1	1
2	3	2	2
3	10	5	3
4	35	13	6
5	126	35	11
6	462	96	21
7	1 716	267	40
8	6 435	750	77
9	24 310	2 123	149
10	92 378	6 046	289
11		17 303	563
12		49 721	1 099
13		143 365	2 152
14		414 584	4 222
15		1 201 917	8 299
16		3 492 117	16 339
17		10 165 779	32 217
18		29 643 870	63 612
19		86 574 831	125 753
20		253 188 111	248 870
21		741 365 049	493 015
22		2 173 243 128	977 576
23		6 377 181 825	1 940 042
24		18 730 782 252	3 853 117
25		55 062 586 341	7 658 211
26		161 995 031 226	15 231 219
27		476 941 691 177	30 312 012
28		1 405 155 255 055	60 360 046
29		4 142 457 992 363	120 260 317
30		12 219 350 698 880	239 727 623
31		36 064 309 311 811	478 105 086
32		106 495 542 464 222	953 950 878
33		314 626 865 716 275	1 904 209 707
34			3 802 587 910
35			7 596 437 240
36			15 180 921 041
37			30 348 394 157
38			60 689 739 010
39			121 403 119 626
40			242 925 445 980
41			486 226 668 328
42			973 467 761 968
43			1 949 468 395 563
44			3 904 970 715 501
45			7 823 872 468 948
46			15 679 198 951 587
47			31 428 242 462 299
48			63 009 591 480 990

For the square lattice, we obtain

$$N_r^s = \frac{1}{2\pi} \int_0^{2\pi} dk\ (1+e^{ik})\, e^{-irk}\, (1+2\cos k)^{s-1} \tag{13}$$

and for the triangular lattice

$$N_r^s = \frac{1}{2\pi} \int_0^{2\pi} dk\ (1+e^{ik})\, e^{-irk}\, (2+2\cos k)^{s-1}. \tag{14}$$

Putting $r = 1$ in equations (13) and (14), we obtain

Square lattice: $$A(s) = (s-1)! \sum_{q=0}^{[s/2]} \frac{(s-q)}{(q!)^2(s-2q)!}, \tag{15}$$

Triangular lattice: $$A(s) = {}^{2s-1}C_s. \tag{16}$$

Equations (15) and (16) check against all the entries in table 1, and it seems reasonable to conjecture that they hold for all s. This would imply that for the square and triangular lattices λ is exactly 3 and 4 respectively, and $\theta = \frac{1}{2}$.

For the hexagonal lattice, we have not been able to guess a closed form expression for $A(s)$. However, guided by the results of square and triangular lattices, we tried a four-parameter fit to $\log A(s)$ of the form

$$\log A(s) \sim s \log \lambda + a - \tfrac{1}{2} \log s + b/s + c/s^2. \tag{17}$$

For s lying between 20 and 48, we find a good fit for $\lambda = 2.0252 \pm 0.0005$. The estimate of error is based on the different values of λ obtained by fitting the form (17) to different ranges of s, and also by trying other asymptotically equivalent functional forms, e.g.

$$\log A(s) \sim s \log \lambda + a - \tfrac{1}{2} \log(s+b) + c/s^2. \tag{18}$$

The simplicity of equation (13) and equation (14) suggests that the two-dimensional directed site animal problem is exactly soluble. It seems likely that the solution would involve proving the interesting and non-trivial combinatorial relations (11) and (12).

Note added in proof. In a recent preprint Redner and Yang (1982) have studied directed bond animals on hypercubical lattices in two to eight dimensions. Their results are consistent with the value $\theta = \frac{1}{2}$ for two-dimensional directed bond animals also.

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